6. R. Christensen, Introduction to the Mechanics of Composites [Russian translation], Mir, Moscow (1982).
7. L. P. Khoroshun, "Toward a theory of saturated porous media," Prikl. Mekh., 12, No. 4 (1976).
8. R. Crock, "Inorganic powder composites," in: Modern Composite Materials [Russian translation], Mir, Moscow (1970).
9. I. F. Martynova and V. V. Skorokhod, "Consolidation of a porous metal during volumetric plastic deformation in the absence of strain-hardening," Poroshk. Metall., No. 5 (1976).
10. I. D. Radomysel'skii, S. V. Titarenko, and N. I. Shcherban', "Effect of a second component on the compressibility of metallic mixtures," in: Theory and Practice of Powder Pressing [in Russian], Inst. Problem Materialovedeniya AN Ukrainian SSR, Kiev (1975).
11. G. S. Pisarenko, V. T. Troshchenko, and A. Ya. Krasovskii, "Study of the mechanical properties of porous iron in tension and torsion," Poroshk. Metall., No. 7 (1965).
12. G. V. Samsonov (ed.), Properties of the Elements: Handbook. Part 1. Physical Properties [in Russian], Metallurgiya, Moscow (1976).

# MECHANICS OF FRACTURE OF COATINGS AND FILMS

A. G. Cherepanov and G. P. Cherepanov

UDC 539.375:621.01

Introduction. The thin surface layer of any material has specific properties which can be attributed to interaction of the material with its environment. The nature of this interaction may be chemical, thermal, or physicomechanical in nature. As a rule, an initial fatigue crack is initiated in the surface layer of materials and structures, and the growth of this crack ultimately leads to exhaustion of the safe life of the structure or structural element. It is therefore natural that the condition of the surface layer (and the control of its properties) is one of the foremost problems currently occupying technologists and engineers [1, 2]. The efficient utilization of materials in industry depends in large part on the solution of this problem.

The surface layer and the body of a material can be regarded together as a composite, one of the components of which is the surface layer [3]. The most important characteristics of this layer are its special protective properties, which depend on its chemical composition and microstructure, the adhesive strength and crack resistance of the layer proper, and the contact of the layer with the substrate. The properties of the surface layer are affected by hydrogen and corrosion in gaseous media and aqueous solutions, wear, catalysis, welding and soldering, erosion, passivation, adhesion, sintering and ablation, and the presence of inhibitors.

Various methods are used to control the mechanical, chemical, magnetic, electrical and other properties of the surface layer. These methods can be classified as follows.

Mechanical Methods. This class of methods includes shot-blasting (to work-harden the surface), hammering, and impact strain-hardening. These methods produce high compressive residual stresses in the surface layer and retard crack nuclei in the layer.

Lacquer Coating and Oxide Films. Lacquer coatings and oxide films serve as chemical protection for the material from the effects of the environment.

Deposition Methods. Deposition methods make it possible to obtain new surface layers with a composition and microstructure different from the composition and microstructure of the substrate material. This class of methods includes plasma deposition, ion deposition, chemical and physical deposition from vapors, and electrolytic deposition.

Methods of Chemico-Physical Modification. Methods of chemico-physical modification of a material make it possible to change the mechanical and physicochemical properties of the surface layer. This class of methods includes special heat treatments, ion nitriding and cementation, ion implantation, and treatment with laser and electron beams. Under natural conditions, metal is usually protected by an oxide film.

---

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 130-140, July-August, 1988. Original article submitted April 10, 1987.

 0021-8944/88/2904-0574$12.50 

Glass Etching. Etching of glass leads to dissolution of a crack-saturated surface layer on a glass object. The layer dissolves into the body of the glass. This process significantly improves glass strength (albeit only for a certain period of time — until new cracks are nucleated in the surface).

The most promising methods are ion implantation and laser treatment.

Ion Implantation. This process can be used to introduce almost any element into the surface layer of any solid. The solid is placed in a vacuum chamber. The new element is introduced by means of a beam of high-velocity ions. The energy of the beam particles is on the order of 0.1-10 MeV. The ions are introduced into the material to a depth on the order of 0.01-1 μm. This method of surface-alloying metals opens up nearly limitless possibilities for forming unusual metastable structures and compounds in the surface layer — structures and compounds which could normally not be formed, due to the thermodynamics of equilibrium and slightly nonequilibrium processes.

Laser Treatment. This process involves the rapid scanning of a surface by a continuous or pulsed laser beam. The beam causes local melting of the thin surface layer, which then cools extremely rapidly due to the tight contact with the mass of cold material at the interface. Here, the thickness of the layer may vary within the range $10^{-1}$-$10^{3}$ μm, while the rate of its cooling reaches $10^{8}$-$10^{10}$ K/sec. Given these cooling rates, metastable ultramicrocrystalline and amorphous structures can form in the surface. These structures are characterized by high strength and exceptionally high fatigue and corrosion resistance, due to the absence of crack nuclei and dislocations in amorphous metals.

Wide use is also being made of combination methods which produce a surface layer with a complex, multilayered structure. An example of this approach would be cold-working of the surface by shot-blasting, followed by anodizing, application of an epoxy primer, and application of a lacquer surface coating. The result here would be a four-layer surface layer.

We will describe the solution of several problems which illustrate the main problems encountered in the fracture mechanics of coatings. The concept of a coating is equivalent to the concept of a surface layer or film.

Optimization of Coatings. Under the influence of alternating loads applied over a period of time, a fatigue-type transverse edge crack develops in a surface layer. This crack may grow into the base metal and lead to premature failure of the structural element. We will derive the necessary equal-strength condition that will allow us to forestall such an event through optimum selection of the fracture toughness of the layer, base metal, and interface.

Let a normal-rupture fatigue crack cross an entire layer of thickness h and reach the interface between the layer $\ell$ and the base metal g (Fig. 1) (it is assumed that plane-stress or plane-strain conditions prevail). At this moment, the material in the region of the crack tip (point O in Fig. 1) is generally nonuniform in regard to its elastic and strength properties. It should be noted that, in practice, it is quite often possible to ignore the nonuniformity of the elastic properties (but not the strength properties!), assuming here that there is a negligible difference between the Young's modulus and Poisson's ratio of the coating and base metal. This assumption is valid for most deposited coatings and cold-worked and modified layers. Thus, uniformity will henceforth be assumed to exist.

First we need to calculate the elastic stresses in a body with a coating but without a crack. We then examine the problem with a crack (Fig. 1). Here, stresses are applied to the edges of the crack. The stresses are equal in magnitude but opposite in sign to the corresponding stresses in the first problem. The local stress field near the end of the crack is determined from the solution of the second problem.

In most cases of practical importance, it can be assumed that the normal load applied to the crack edges in the second problem is constant and equal to $\sigma_y = -p = -E_l \varepsilon_y (1 - \nu_l^2)^{-1}$, while the shear load is equal to zero for the free boundary (Fig. 1). Here, $E_\ell$ is the Young's modulus of the coating, $\varepsilon_y$ is the strain at the point of the crack's location obtained from the solution of the first problem in the simplest case $h \to 0$, $E_\ell = E_g$ ($E_g$ is the Young's modulus of the substrate). This result can be rigorously proven for almost any thin coating when $h/R \ll 1$ and $E_\ell \sim E_g$ (or $E_\ell \ll E_g$) (R is the radius of curvature of the surface at the point under consideration). At $E_\ell \gg E_g$, due to the presence of two small parameters $E_g/E_\ell$

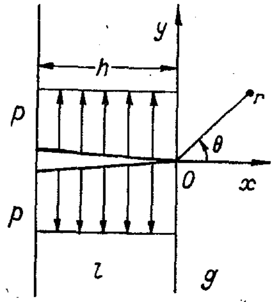

Fig. 1

and h/R, under certain conditions it may be necessary to account for the bending of the shell (i.e., the term in the load on the crack which is linear with respect to the coordinate). However, such stiff protective coatings are almost never used, due to the high stress concentrations in them. For metals, such coatings do not exist in nature (the Young's modulus of the most rigid material – diamond – is only six times greater than that of iron).

We will introduce polar coordinates $r\theta$ with the center at the crack tip (Fig. 1). We will examine a small region around the crack tip ($r \ll h$). A fatigue crack can travel from point 0 along three possible paths (Fig. 2), depending on the relation describing the strength nonuniformity: a) in the old direction $\theta = 0$, but in the substrate; b) along the interface between the substrate and the coating $\theta = \pm\pi/2$; c) in the material of the layer, near the interface $\theta \approx \pm\pi/2$. Other conditions being equal, the length of reliable service of the structure will obviously be shorter in the first case than in the second and third cases – when the interface slows the crack. In evaluating the safe life of a structure at the design stage, it is absolutely mandatory that this circumstance be taken into account by intentional alteration of the properties of the coating and the interface in the appropriate direction. It is important to emphasize that, as a rule, excessively strong coatings lead to a reduction in service life.

On the whole, the optimum coating is that coating which maximizes service life. The service life is determined by the sum of four terms: the period to the nucleation of a fatigue crack; the period of growth of the crack through the entire thickness of the coating; the period during which the crack is held at the interface; the period of subcritical growth in the substrate. Each of these periods must be studied independently, since, under certain conditions, any one of them may prove to be decisive in the choice of the optimum coating. We will examine the third period in greater detail.

The stress $\sigma_\theta$ near the crack tip has the form [5]

$$\sigma_\theta = K_I(\lambda + 1)(2\pi)^{-1/2} r^\lambda [(2+\lambda)\cos\lambda\theta + B\cos(\lambda+2)\theta]$$
$$(0 \leqslant \theta \leqslant \pi/2),$$
$$B = \frac{k_1(3\lambda+2) - k_2(1+2\lambda) + \lambda + 1}{1+k_1}, \tag{1}$$
$$k_1 = \frac{k-1}{4(1-\nu_l)}, \quad k_2 = \frac{1-\nu_g}{1-\nu_l}k, \quad k = \frac{\mu_l}{\mu_g},$$
$$K_I = \eta(k, \nu_l, \nu_g)p(\pi h)^{-\lambda},$$

where $\nu$ and $\mu$ are the Poisson's ratio and shear modulus (the subscripts $\ell$ and g denote the coating and the substrate, respectively); $K_I$ is the stress intensity factor in the given problem (see Fig. 1); $\eta(k, \nu_\ell, \nu_g)$ is a function of elasticity theory which is determined by numerical methods (its graph from [4] is shown in Fig. 3 with $\nu_\ell = \nu_g = 0.3$); $\lambda$ is the unique real root of the equation

$$\cos\pi\lambda = a + b(\lambda+1)^2$$
$$\left(a = \frac{2k_1^2 - 2k_1k_2 + 2k_1 - k_2 + 1}{2(k_2-k_1)(k_1+1)}, \quad b = \frac{2k_1}{k_1+1}\right),$$

lying in the interval (–1, 0). The dependence of this root on k at $\nu_\ell = \nu_g = 0.3$, taken from [5], is shown in Fig. 4. In the most important special case of an isotropic homogeneous body, when $\nu_\ell = \nu_g$, $\mu_\ell = \mu_g$, we have

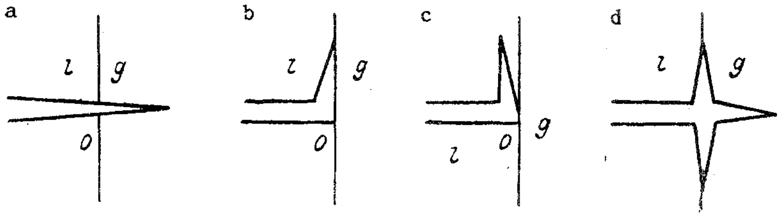


Fig. 2

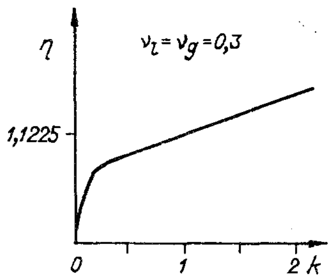


Fig. 3

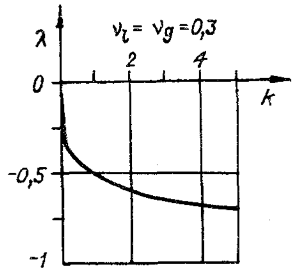


Fig. 4

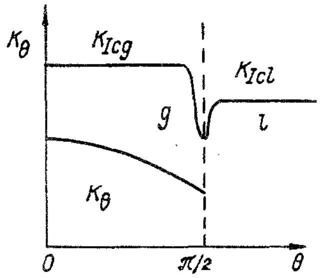


Fig. 5

$$\sigma_\theta = \frac{1}{4} K_I (2\pi r)^{-1/2} \left(3 \cos \frac{1}{2}\theta + \cos \frac{3}{2}\theta\right), \quad K_I = 1.1225 p (\pi h)^{1/2}. \tag{2}$$

We will use $K_\theta(\theta)$ to represent the function $K_\theta = \sigma_\theta (2\pi)^{1/2} r^{-\lambda}$ determined by Eq. (1). It can be seen that this function is monotonically decreasing (Fig. 5). The graph of the dependence of fracture toughness $K_{Ic}$ on θ is shown qualitatively, assuming here that the interface is the weakest element.

First we will examine monotonic loading, when the load p increases monotonically over time. Then, in accordance with the theory of generalized normal rupture [4] (which has been substantiated empirically), a crack in the direction $\theta = \theta_*$ in which $K_\theta(\theta_*) = K_{Ic}(\theta_*)$, while in all of the other directions $\theta \neq \theta_*$, $K_\theta(\theta) < K_{Ic}(\theta)$. If the fracture toughnesses of the coating and interface are lower than the fracture toughness of the substrate, then the curve $K_\theta(\theta)$ first intersects the curve $K_c(\theta)$ at $\theta = 0$ or $\theta = \pi/2$ (Fig. 5), i.e., one of the three cases discussed above is realized.

We will refer to the coating as being of equal strength with respect to the limit load if its properties are such that the curve $K_\theta(\theta)$ initially touches the curve $K_{Ic}(\theta)$ simultaneously at the points $\theta = 0$ and $\theta = \pi/2$ [here, the crack moves simultaneously in both directions, with the formation of retarding "shear" (see Fig. 2d)]. In accordance with (1), we have

$$\begin{aligned} &\text{at } \theta = 0 \;\; K_\theta = K_I(\lambda + 1)(\lambda + 2 + B), \\ &\text{at } \theta = \frac{\pi}{2} \;\; K_\theta = K_I(\lambda + 1)\left[(2 + \lambda)\cos\frac{\pi\lambda}{2} + B\cos\frac{\pi}{2}(\lambda + 2)\right]. \end{aligned} \tag{3}$$

For a homogeneous body, in accordance with (2), we have

$$\text{at } \theta = 0 \;\; K_\theta = K_I, \quad \text{at } \theta = \pi/2 \;\; K_\theta = K_I/(2\sqrt{2}). \tag{4}$$

We use (3) to derive the following equal-strength condition:

at $K_{Icgl} < K_{Icl}$

$$K_I(\lambda + 1)(\lambda + 2 + B) = K_{Icg},$$
$$K_I(\lambda + 1)\left[(2 + \lambda)\cos\frac{\pi\lambda}{2} + B\cos\frac{\pi(\lambda + 2)}{2}\right] = K_{Icgl},$$

i.e.,

$$(\lambda + 2 + B) K_{Icgl} = K_{Icg}\left[(2 + \lambda)\cos\frac{\pi\lambda}{2} + B\cos\frac{\pi(\lambda + 2)}{2}\right]; \tag{5}$$

at $K_{Icgl} > K_{Icl}$

$$(\lambda + 2 + B) K_{Icl} = K_{Icg}\left[(2 + \lambda)\cos\frac{\pi\lambda}{2} + B\cos\frac{\pi(\lambda + 2)}{2}\right]. \tag{6}$$

Here, $K_{Icg\ell}$ is the fracture-toughness of the interface, having the dimension of force divided by the length to the power $2 + \lambda$; $K_{Icg}$ and $K_{Ic\ell}$ are the fracture toughnesses of the substrate and the layer.

In accordance with (4), we have the following equal-strength condition for a material which is uniform with respect to its elastic properties (but not its strength)

$$2\sqrt{2}\min(K_{Icgl}, K_{Icl}) = K_{Icg}\ (\lambda = -1/2, \nu_l = \nu_g, E_l = E_g). \tag{7}$$

Thus, for an equal-strength coating during monotonic loading, the fracture toughness of the substrate should be $2\sqrt{2}$ times greater than the fracture toughness of the interface (if $K_{Icg\ell} < K_{Ic\ell}$) or the coating (if $K_{Icg\ell} > K_{Ic\ell}$). The values of $K_{Icg\ell}$ and $K_{Ic\ell}$ can be controlled by resorting to various surface treatments. The coating becomes a protective layer (protecting the material from cracks under limit loads) if the equal sign in equal-strength conditions (5)-(7) can be replaced by the sign <.

Let us examine cyclic loading, when the load p is a periodic function of time. Let a fatigue crack grow from the point 0 – the crack tip at the interface – at a certain angle $\theta$ (see Fig. 1). In this case, the fatigue-crack growth rate $d\ell/dN$ will be a certain function of the maximum and minimum values of $K_\theta$ over a cycle $d\ell/N = f(K_{\theta max}, K_{\theta min}, \theta)$. In the general case, the rate will also depend on the angle $\theta$. It is natural to propose that the direction of growth of the main fatigue crack is given by the angle $\theta = \theta_*$ for which the function f reaches its maximum

$$\max_\theta f(K_{\theta\max}(\theta), K_{\theta\min}(\theta), \theta) \text{ for } \theta = \theta_*. \tag{8}$$

In the other directions, fatigue cracks enter the unloading region and stop growing.

In the case of two different homogeneous media, only the above three variants a-c are in competition at point O, while the condition for selection of the direction (8) takes the form $d\ell/dN = \max\{f_g, f_{\ell g}, f_\ell\}$ ($f_g$, $f_{\ell g}$, and $f_\ell$ are the rates of growth of the fatigue crack in the directions $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi/2 + 0$, respectively). These quantities are determined experimentally. If the difference in the elastic constants of the two media can be ignored, then the values of $f_g$ and $f_\ell$ can be taken from the curves $d\ell/dN - \Delta K$ for the corresponding homogeneous material with $\Delta K$ determined from Eqs. (1). Here, $f_{\ell g}$ is found from the analogous curve recorded for a fatigue crack growing along the interface.

The above remarks make it obvious that the directions of growth will generally be different for curvilinear fatigue cracks growing under low alternating loads and equilibrium cracks growing under limit loads. This is one consequence of the general result that the behavior of materials and structures is different under large and small loads: a structure might withstand large limit loads but have a relatively short life under low loads, and vice versa.

In connection with the above finding, it is natural to consider a coating to be equal-strength with respect to fatigue life if the directions $\theta = 0$ and $\theta = \pi/2$ are equivalent, i.e.,

$$f_g = \max(f_l, f_{lg}). \tag{9}$$

If

$$f_g < \max(f_l, f_{lg}), \tag{10}$$

then the coating can be considered protective against fatigue cracks.

It follows from this that given a durable coating and substrate (i.e., with values of $f_\ell$ and $f_g$ satisfying the inequality $f_g > f_\ell$), a designer should choose the technology for forming the bond between the coating and the substrate so as to satisfy the condition $f_{\ell g} > f_g$ at $f_g > f_\ell$, guaranteeing that the coating will help protect against fatigue cracks. At the same time, it is also necessary to ensure that the strength of the bond is adequate under limit loads [3]. It should be emphasized that conditions (9) and (10) include the working load and the thickness of the layer; thus, the coating and its bond with the substrate should be designed for a certain level of cyclic or variable loading (and for a specified safe life). Even in the case of the simplest power relation $\Delta\ell/dN \sim (\Delta K)^m$, p ceases to be dependent on h only when the constant m is the same for the substrate and the interface.

When the properties of the surface layer change smoothly with depth and there is no interface, the coating does not protect against fatigue cracks because the crack always moves

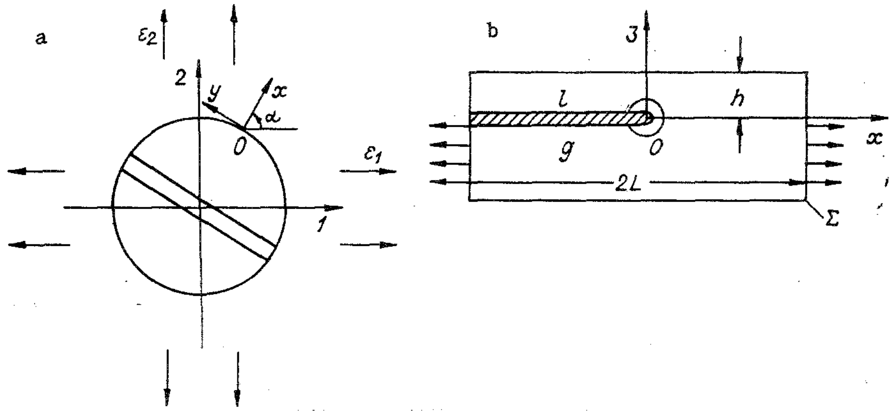


Fig. 6

into the base metal without stopping. In this case, it is necessary to strive to provide for orthotropy in the surface layer in regard to the growth of longitudinal and transverse fatigue cracks. Here, as before, the necessary condition for the slowing of transverse fatigue cracks is expressed by inequality (10), where ℓg is the longitudinal direction in the layer.

Fatigue-Crack Growth along the Coating–Substrate Interface. Let a plane region in which bonding between the coating and substrate is absent be located on the coating–substrate interface (the initial crack, shown in plan in Fig. 6a). This crack may have developed either as a result of processing or as a consequence of retardation of a transverse fatigue crack in the layer at the interface by the above-described mechanism. We will assume that the characteristic linear dimension of the crack in plan is much greater than the thickness of the coating.

We will also assume that the stress–strain state of the part at the site being examined (without allowance for the crack and coating) reduces to the principal strains $\varepsilon_1$ and $\varepsilon_2$ and the stresses:

$$\sigma_1 = \frac{E_\delta}{1-\nu_\delta^2}(\varepsilon_1 + \nu_\delta\varepsilon_2),\ \sigma_2 = \frac{E_\delta}{1-\nu_\delta^2}(\varepsilon_2 + \nu_\delta\varepsilon_1)$$
$$(\delta = l,\ g;\ E_\delta\varepsilon_3 = -\nu_\delta(\sigma_1 + \sigma_2)).$$

The direction 3 coincides with a normal to the free surface of the body.

We suppose that $\varepsilon_1$ and $\varepsilon_2$ include not only the strains which develop as a result of the service loads, but also large-scale (compared to h) plastic and thermal strains. It is easy to see that if the coating over the crack is continuous, then no stress concentration will develop at the crack front. Thus, the force driving the crack and expressed by the invariant Γ-integral will be equal to zero. Here, the contour of the crack is stationary. If the coating has through transverse cracks or scratches, then the Γ-residue at the crack front will be nontrivial and the crack will move [4, 6].

We will restrict ourselves to the most concrete and important case, when the system of through cracks is such that it reduces all of the stresses in the region of the coating above the crack to zero (except for a narrow strip on the order of the coating thickness along the border of the crack, where there are three-dimensional stress distributions resulting from the external field).

Let us examine the neighborhood of an arbitrary point O on the crack front. The neighborhood is small compared to the radius of curvature of the crack contour at this point but is large compared to the thickness of the coating (Fig. 6b). We will use x and y to denote the normal and tangent to the contour of the crack at this point.

In the plane problem in Fig. 6b, the stresses in the strip $0 < y < h$ approach zero as $x \to -\infty$. In the remainder of the region, as $x \to \pm\infty$, the stresses and strains approach their unperturbed values

$$\varepsilon_x = \varepsilon_1\cos^2\alpha + \varepsilon_2\sin^2\alpha,\ \varepsilon_y = \varepsilon_1\sin^2\alpha + \varepsilon_2\cos^2\alpha,$$
$$\varepsilon_{xy} = (1/2)(\varepsilon_2 - \varepsilon_1)\sin 2\alpha \quad (11)$$

(the stresses are determined from Hooke's law). Here, α is the angle made by the normal to the contour of the crack with the direction $\varepsilon_1$.

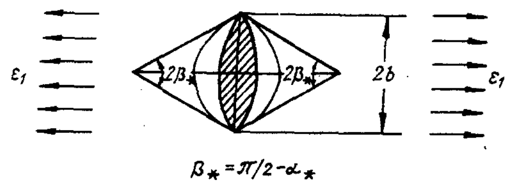


Fig. 7

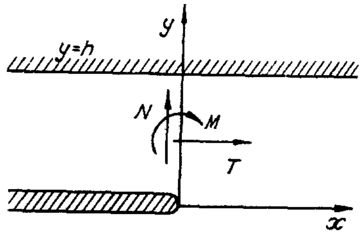


Fig. 8

We will calculate the Γ-residue over the small neighborhood enveloping the point O. To do this, we take the closed contour Σ, consisting of the free edges of the crack, the free surface of the body, the lines $x = \pm L_1$ at $L \to \infty$, and lines parallel to the x-axis and removed at distance $L_2$ from this axis at $L_2 \to \infty$ (rectangle with a slit − see Fig. 6b). In practice, it suffices to take $L_1$ and $L_2$ to be on the order of 3h. Using the theory of Γ-integration [3, 4], we obtain

$$\Gamma = \oint_{\Sigma}\left[\left(Z + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\right)n_x - \sigma_{ij}u_{i,x}n_j\right]d\Sigma = -h\left(Z_\infty - \sigma_x\varepsilon_x - 2\tau_{xy}\varepsilon_{xy}\right) = -\frac{1}{2}h\left(\sigma_y\varepsilon_y - \sigma_x\varepsilon_x - 2\tau_{xy}\varepsilon_{xy}\right), \quad (12)$$

where Z is the strain energy of a unit volume; ρ is density; $\dot{u}_i$ is velocity; $u_i$ represents displacements; $\sigma_{ij}$ represents stresses; the stresses with the subscripts x and y and the quantity $Z_\infty$ pertain to the unperturbed state of the coating (11); Γ is the force driving the crack [4, 6]. Thus, in the general case (when the plastic region is large), the rate of growth of a fatigue crack $d\ell/dN$ is expressed by the formula [3, 4]

$$dl/dN = f(\Gamma_{max}, \Gamma_{min}) \quad (13)$$

(f is a certain function; $\Gamma_{max}$ and $\Gamma_{min}$ are the maximum and minimum values of Γ per cycle).

Using the similarity postulate and the general energy concept for small plastic regions − whereby it is possible to employ the notion of a stress intensity factor − we find [4]

$$\frac{dl}{dN} = -\beta\left(\frac{\Gamma_{max} - \Gamma_{min}}{\Gamma_c} + \ln\frac{\Gamma_c - \Gamma_{max}}{\Gamma_c - \Gamma_{min}}\right) \quad (\Gamma_c \geqslant \Gamma_{max} > \Gamma_Y)$$

(β, $\Gamma_c$, $\Gamma_Y$ are adhesion constants of the coating−substrate interface). In accordance with (11)-(13), the rate of growth of the fatigue crack is independent of its dimensions and the number of cycles, but it depends considerably on the angle α (and, of course, on the periodic external load).

Let the equation of the contour of a moving crack have the form f(x, y, t) = 0 (t is the time, measured in cycles). We introduce the following differential equation for the function f:

$$v_x\partial_x f + v_y\partial_y f + \partial_t f = 0. \quad (14)$$

Here, $v_x$ and $v_y$ are components of the velocity of the crack contour at the point being examined. These components are determined from the formulas $v_x = v(\alpha)\cos\alpha$, $v_y = v(\alpha)\sin\alpha$, $v(\alpha) = d\ell/dN$, $\tan\alpha = \partial_y f/\partial_x f$ (v is the normal velocity of the fatigue-crack contour).

Thus, the motion of the contour of the fatigue crack is found from the solution of a first-order nonlinear differential equation in partial derivatives. This equation can easily be solved numerically by using the time-step method.

In the simplest case of uniaxial tension, when $\varepsilon_2 = 0$, we calculate

$$\Gamma = \frac{hE_l\varepsilon_1^2\cos^2\alpha}{2(1 - \nu_l^2)}\left[1 - \mathrm{tg}^2\alpha + 2(1 - \nu_l)\sin^2\alpha\right]. \quad (15)$$

For example, let the initial crack be a narrow ellipse of the length 2b elongated along the direction of tension $\varepsilon_1$ (the hatched region in Fig. 7). The fatigue crack begins to move during cyclic tension; here, in accordance with (15), the central region will move forward at a greater velocity than the edge, while the crack tips will remain stationary. If we consider that $\Gamma_Y = 0$, then the crack will continue to move until the driving force Γ is greater than zero over the entire contour of the crack. When Γ = 0, motion of the crack ceases. Thus, the condition Γ = 0 makes it possible to directly determine the limiting contour of the crack without solving the differential equation. It follows from this and (15) that

$$\mathrm{tg}^2 \alpha_* = 1 + 2(1 - \nu_l) \sin^2 \alpha_*. \quad (16)$$

Thus, the limiting contour of the crack is a rhombus with the diagonal 2b and the divergence angle $2\alpha_*$, being a root of Eq. (16). For example, $\alpha_* = 52°$ with $\nu_\ell = 0.5$, $\alpha_* = 55°$ with $\nu_\ell = 0.3$, and $\alpha_* = 58°$ with $\nu_\ell = 0$. It is interesting that the limiting contour is formed after a finite number of cycles $N_*$. This number is easily found if we divide the crack velocity into the path, of length $b \sin \alpha_*$, which is traversed by the central point of the contour along the diameter of the rhombus (with $\alpha = 0$). For example, in the simplest case, when $\varepsilon_{1\,\min} = 0$, $dl/dN = A\Gamma^2_{\max}$, we can use (15) to find $N_* = 4b(1 - \nu_l^2)^2 \sin \alpha_* / (Ah^2 E_l^2 \varepsilon^4_{1\max})$.

When $\Gamma_Y > 0$, the limiting contour will be inside the rhombus. In the case of an initial crack of arbitrary contour, it is now easy to construct the limiting contour. We draw tangents to the initial contour at the angle $\pi/2 - \alpha_*$ with the tensile direction.

Structure of a Delamination Crack. The question of the stress and strain distribution near the front of a delamination crack with allowance for the surface of the body is fairly complex in a computational sense, since it requires analysis of exact equations describing the deformation of the material at distances on the order of the thickness of the coating. Proceeding on the basis of the "microscope principle," we will study the region in this case as a half-plane with a semi-infinite slit parallel to the boundary. Plane strain conditions prevail in the region [3]. Figure 8 shows the region along with the resultants of the force and moment, referred to the coordinate origin and the crack tip.

We will examine this problem in the simplest linear-elastic formulation, assuming that the elastic constants of the coating and substrate are identical. The exact solution of this problem was obtained in [7] by an elegant but very laborious analytical method using a Riemannian matrix problem. Thus, the homogeneous isotropic half-plane $y < h$ has a semi-infinite slit at $y = 0$, $x < 0$. The slit is parallel to the boundary of the half-plane. The edges of the slit and the boundary $y = h$ are assumed to be free of loads.

The resultant of the stresses $\sigma_y$ and $\tau_{xy}$ in the section $y = 0$ is equal to the forces N and T, while their moment relative to the origin is equal to M.

Let us present the final results of the solution for the stress intensity factors [7]

$$K_{\mathrm{I}} = 1.932M/h^{3/2} - 0.5314T/\sqrt{h} + 1.933N/\sqrt{h},$$
$$K_{\mathrm{II}} = -1.506M/h^{3/2} + 1.3108T/\sqrt{h} + 0.033N/\sqrt{h}. \quad (17)$$

We check them by means of the invariant Γ-integral.

The forces N and T and the moment M in the given problem are created by the following stress state in the strip $0 < y < h$ at $x \to -\infty$:

$$\sigma_x = 2(3M - 2hT)h^{-2} + 6y(hT - 2M)h^{-3} + 6xNh^{-3}(2y - h),\ \sigma_y = 0,\ \tau_{xy} = -6Nyh^{-3}(y - h). \quad (18)$$

We calculate the Γ-residue at the crack tip for the given problem, using the contour in Fig. 6b in a simple calculation [3, 4] similar to (12). We will present the intermediate calculations for the displacements at $0 < y < h$ and $x \to -\infty$

$$u = \frac{1-\nu^2}{E} 2xh^{-2}\left[3M - 2hT + 3yh^{-1}(hT - 2M)\right] +$$
$$+ \frac{1+\nu}{E} Nh^{-3}\left[3(1-\nu)x^2(2y - h) + 2y^2(\nu - 2)\left(y - \frac{3}{2}h\right)\right] + C_1,$$
$$v = \frac{1+\nu}{E}\{2\nu yh^{-2}(3M - 2hT) - 3h^{-3}(hT - 2M)[\nu y^2 + (1-\nu)x^2]\} +$$
$$+ \frac{1+\nu}{E} Nh^{-3}[2x^3(1-\nu) - 6\nu xy(y - h)] + C_2 \quad (19)$$
$$(E\varepsilon_x = (1-\nu^2)\sigma_x,\ E\varepsilon_y = -\nu(1+\nu)\sigma_x,\ E\varepsilon_{xy} = (1+\nu)\tau_{xy},\ \varepsilon_z = 0)$$

($C_1$ and $C_2$ are nonessential constants).

Equations (18) and (19) constitute the exact solution of the equations of elasticity theory for the strip $0 < y < h$ with the free boundary $\sigma_y = \tau_{xy} = 0$ at $y = 0$, $y = h$ and with specified values of M, T, and N in the section $x = 0$.

Thus,

$$\Gamma = \int_0^h \left(\frac{1}{2}\sigma_x\varepsilon_x + \tau_{xy}\varepsilon_{xy} - \tau_{xy}\frac{\partial u}{\partial y}\right) dy = \frac{1-\nu^2}{E}\left(6M^2h^{-3} - 6MTh^{-2} + 2T^2h^{-1} + \frac{6}{5}N^2h^{-1}\right). \quad (20)$$

Equation (20) is also valid for a plane stress state if we put $\nu = 0$ in the equation.

Let us attempt to calculate the stress intensity factors from the known value of the Γ-residue. We will restrict ourselves to the case of three-independent loading parameters $p_1$, $p_2$, $p_3$ (as in the given problem). In accordance with the superposition principle, we have $K_I = \alpha_i p_i$, $K_{II} = \beta_i p_i$ ($i = 1, 2, 3$) ($\alpha_i$ and $\beta_i$ are sought coefficients which are independent of $p_i$). We then find

$$\Gamma = \frac{1-\nu^2}{E}(K_I^2 + K_{II}^2) = \frac{1-\nu^2}{E}(\alpha_i\alpha_k + \beta_i\beta_k)\,p_i p_k = \frac{1-\nu^2}{E}A_{ik}p_i p_k \ (i, k = 1, 2, 3) \tag{21}$$

($A_{ik}$ are known coefficients). Thus, we obtain six equations to determine six unknowns:

$$\alpha_i\alpha_k + \beta_i\beta_k = A_{ik} \ (i,\ k = 1, 2, 3). \tag{22}$$

We write the solution of this system of equations in the form

$$\alpha_i = \sqrt{a_k}\sin\gamma_k,\ \ \beta_i = \sqrt{a_k}\cos\gamma_k,$$

$$\gamma_1 = \gamma_3 + \arccos\frac{A_{13}}{\sqrt{A_{11}A_{33}}},\ \ \gamma_2 = \gamma_3 + \arccos\frac{A_{23}}{\sqrt{A_{22}A_{33}}}$$

$$(a_1 = A_{11},\ a_2 = A_{22},\ a_3 = A_{33}),$$

since the following condition must be satisfied for system (22) to be solvable:

$$\arccos\frac{A_{13}}{\sqrt{A_{11}A_{33}}} - \arccos\frac{A_{23}}{\sqrt{A_{22}A_{33}}} = \arccos\frac{A_{12}}{\sqrt{A_{11}A_{22}}}. \tag{23}$$

Thus, all of the stress intensity factors can be determined from Γ with an accuracy up to one unknown constant. Also, it can be shown that this result is valid for any number of load parameters. Thus, in problems with a known value of Γ, to find the coefficients $K_I$ and $K_{II}$, it is sufficient to determine just one value $K_I$ or $K_{II}$ with any one value of one of the load parameters (i.e., one of the coefficients $\alpha_i$ or $\beta_i$).

It is not hard to see that, in accordance with (17), the coefficients $A_{ik}$ in Eq. (21) of the given problem do not satisfy condition (23). This is not the result of some elementary computing error. The reason is nontrivial, and to explain it we will look closer at the problem we have been considering (see Fig. 8), with $M = T = 0$, $N \neq 0$. We will refer to this as problem N. In accordance with (18), the stress $\sigma_x$ in the strip $0 < y < h$ approaches infinity as $x \to -\infty$. Meanwhile, in each section of the strip $x = \text{const} \to -\infty$, $M = \int_0^h y\sigma_x dy = -Nx$, while the resultant force $N \neq 0$, $T = 0$. Thus, judged on the basis of the theorem of correct boundary conditions [4], the N problem is incorrect because one of the necessary conditions for correctness of a boundary-value problem with a cylindrical, infinitely-distant point – boundedness of the stresses at infinity – is not satisfied.

We can construct as many different solutions of the N problem as we like in the class of stresses not bounded at $x \to -\infty$, $0 < y < h$ and giving $M = 0$, $T = 0$, $N = \text{const} \neq 0$ in the section $x = -0$. With $M = 0$, $T = 0$, Eqs. (18) and (19) give one solution from this solution set. The Γ-integral diverges in the neighborhood of the cylindrical, infinitely-distant point in the N problem. Thus, it can take any value, depending on the path of integration and the chosen asymptote [in particular, it is finite and equal to (20) when $M = 0$, $T = 0$ – as in the given case]. The same as true in regard to the stress intensity factors $K_I$ and $K_{II}$ of the N problem: the solution in [7] is incorrect when $M = 0$, $T = 0$, $N \neq 0$.

It should be noted that if the Γ-integration rule [3, 4] is applied to the divergent integral (20) in the N problem, then we obviously obtain $\Gamma = 0$. Since $\Gamma \sim N^2$, it follows that $N = 0$: this is the condition of correctness of the N problem.

## LITERATURE CITED


1. H. Herman (ed.), Treatise on Materials Science and Technology, Vols. 1-20, Acad. Press, N.Y. (1976-1982).
2. Surface Treatment and the Reliability of Materials [Russian translation], Mir, Moscow (1985).
3. G. P. Cherepanov, Mechanics of Fracture of Composite Materials [in Russian], Nauka, Moscow (1983).
4. G. P. Cherepanov, Mechanics of Brittle Fracture, McGraw Hill, N. Y. (1979).

5. A. R. Zak and M. L. Williams, "Crack point stress singularities at bimaterial interface," J. Appl. Mech., 30, No. 1 (1963).
6. G. S. Pisarenko, V. P. Naumenko, and G. S. Volkov, Determination of the Fracture Toughness of Materials on the Basis of the Energy Contour Integral [in Russian], Naukova Dumka, Kiev (1978).
7. A. N. Zlatin and A. A. Khrapkov, "Semi-infinite crack parallel to the boundary of an elastic half-space," Dokl. Akad. Nauk SSSR, 291, No. 4 (1986).

# APPLICATION OF METHODS OF THE MECHANICS OF HETEROGENEOUS MEDIA TO DESCRIBE DISPERSION PROCESSES IN AN ELECTROMAGNETIC FIELD

S. Yu. Arutyunov, I. N. Dorokhov, V. V. Kafarov, V. G. Korniichuk, and V. P. Solov'ev

UDC 538.4:621.926.085.54-185

The authors of [1, 2] found equations describing the thermohydromechanics of a two-phase polydisperse medium which take into account refinement of the particles of the disperse phase. Here, we use the methods of the mechanics of heterogeneous media [3] and fundamental equations of electrodynamics [4] to obtain a mathematical description of the refinement process in an electromagnetic field (EMF) which considers the collision, destruction, and formation of disperse-phase particles and the effect of the EMF on these events.

1. The assumptions used in [1-3] are adopted here to study the motion of a heterogeneous mixture of three phases in an EMF. The first phase is the carrier phase (liquid or gas), while the second and third phases are in the form of individual particles of a material undergoing refinement and bodies of different sizes undergoing fragmentation.

We introduce the volume contents of the phase $\alpha_i$ and the mean densities $\rho_i$ ($i = 1, 2, 3$) at each point of the volume occupied by the mixture:

$$\alpha_1 + \int_0^R r f_2(r)\,dr + \int_{R_1}^{R_2} \mu f_3(\mu)\,d\mu = 1, \quad \rho = \rho_1 + \int_0^R \rho_2^0 r f_2(r)\,dr + \int_{R_1}^{R_2} \rho_3^0 \mu f_3(\mu)\,d\mu.$$

Here, the polydispersity of the second phase is characterized by the function $f_2(r)dr$ [the number of particles undergoing refinement per unit volume whose dimensions (volumes) are within the range $(r, r + dr)$], while the polydispersity of the third phase is characterized by the function $f_3(\mu)d\mu$ [the number of bodies undergoing fragmentation per unit volume whose dimensions (volumes) are within the interval $(\mu, \mu + d\mu)$]; the subscripts 1, 2, and 3 pertain to the carrier phase, the disperse phase, and the phase comprised of bodies being fragmented; $R_1 > R$. Following [1, 2], we introduce the notion of an r-phase as an aggregate of particles whose dimensions lie within the interval $(r, r + dr)$. We also introduce the notion of a μ-phase as an aggregate of fragmenting particles whose dimensions lie within the interval $(\mu, \mu + d\mu)$. Each phase represents a charged, electrically-conducting, polarized, and magnetized medium in the EMF.

2. In constructing models of continua interacting with an EMF, various methods of formulating the equations of electrodynamics can be used – depending on the expressions used for the local field strengths $\mathbf{E}_1, \mathbf{E}_2(r), \mathbf{E}_3(\mu)$ and $\mathbf{H}_1, \mathbf{H}_2(r), \mathbf{H}_3(\mu)$. However, after one of these formulations is chosen, all of the conservation laws of mechanics must be considered with allowance for this formulation. The formulation of the equations of electrodynamics currently in widest use is the Chu model [5]. We assume that

$$\mathbf{E}_1^* + \mu_0\mathbf{H}_1 \times \mathbf{v}_1 = \mathbf{E}_1, \quad \mathbf{H}_1^* - \varepsilon_0\mathbf{E}_1 \times \mathbf{v}_1 = \mathbf{H}_1,$$
$$\mathbf{E}_2^*(r) + \mu_0\mathbf{H}_2(r) \times \mathbf{v}_2(r) = \mathbf{E}_2(r), \quad \mathbf{H}_2^*(r) - \varepsilon_0\mathbf{E}_2(r) \times \mathbf{v}_2(r) = \mathbf{H}_2(r),$$
$$\mathbf{E}_3^*(\mu) + \mu_0\mathbf{H}_3(\mu) \times \mathbf{v}_3(\mu) = \mathbf{E}_3(\mu), \quad \mathbf{H}_3^*(\mu) - \varepsilon_0\mathbf{E}_3(\mu) \times \mathbf{v}_3(\mu) = \mathbf{H}_3(\mu).$$

Then the electrodynamical equations take the form

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 140-145, July-August, 1988. Original article submitted May 6, 1987.

0021-8944/88/2904-0583$12.50  